

Admissible subsets and Littelmann paths in affine Kazhdan-Lusztig theory

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Abstract

The center of an extended affine Hecke algebra is known to be isomorphic to the ring of symmetric functions associated to the underlying finite Weyl group W_0 . The set of Weyl characters s_λ forms a basis of the center and this character acts as translation on the Kazhdan-Lusztig basis element C_{w_0} where w_0 is the longest element of W_0 , that is we have $s_\lambda C_{w_0} = C_{p_\lambda w_0}$. As a consequence, we see that the coefficients that appears when decomposing $s_\tau s_\lambda C_{w_0} = s_\tau C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis are tensor multiplicities of the Lie algebra with Weyl group W_0 . The aim of this paper is to explain how admissible subsets and Littelmann paths, which are models to compute such multiplicities, naturally appear when working out the decomposition of $s_\tau C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis.

1 Introduction

Let W_e be an extended affine Weyl group with underlying finite Weyl group W_0 . Then $W_e = W_0 \ltimes P$ where P denotes the set of weights associated to W_0 . Let \mathcal{H}_e be the associated generic affine Hecke algebra defined over the ring \mathcal{A} of Laurent polynomials with one indeterminate and let $\{C_w \mid w \in W_e\}$ be the Kazhdan-Lusztig basis of \mathcal{H}_e . The center of the affine Hecke algebra \mathcal{H}_e associated to W_e is known to be isomorphic to the ring of symmetric functions $\mathcal{A}[P]^{W_0}$. The set of Weyl characters $\{s_\lambda \mid \lambda \in P^+\}$ forms a basis of $\mathcal{A}[P]^{W_0}$ and we have $s_\lambda C_{w_0} = C_{p_\lambda w_0}$ where p_λ denotes the translation by $\lambda \in P^+$ in W_e ; see [7].

Denote by $V(\tau)$ the irreducible highest weight module of weight $\tau \in P^+$ for the simple Lie algebra over \mathbb{C} with Weyl group W_0 . Then the character of $V(\lambda)$ is s_λ and for all $\tau, \lambda \in P^+$ we have $s_\tau s_\lambda = \sum m_{\tau, \lambda}^\mu s_\mu$ where $m_{\tau, \lambda}^\mu$ is the multiplicity of $V(\mu)$ in the tensor product $V(\tau) \otimes V(\lambda)$. Computing the multiplicities $m_{\tau, \lambda}^\mu$ is one of the most basic question in representation theory of simple Lie algebras over \mathbb{C} . Littelmann showed [10] that such multiplicities can be determined by counting certain kind of paths in the weight lattice P constrained to stay in the fundamental chamber. Later on, Lenart and Postnikov [8, 9] showed that these multiplicities can be determined using admissible subsets associated to a fix reduced expression of $p_\tau \in W_e$. Their model can be viewed as a discrete counterpart of Littelmann paths model and they explicitly constructed a bijection between admissible subsets and Littelmann paths.

In the extended affine algebra, we must have

$$s_\tau C_{p_\lambda w_0} = s_\tau s_\lambda C_{w_0} = \sum m_{\tau, \lambda}^\mu C_{p_\mu w_0}$$

and the aim of this paper is to explain why these multiplicities appear when decomposing $s_\tau C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis. This will be done in two steps:

- 1) we will show that to determine the decomposition of the product $s_\lambda C_{p_\tau w_0}$ it is sufficient to study products of the form $T_{p_\lambda} T_{p_\tau w_0}$ in the standard basis and to find the terms of maximal degrees;
- 2) we will show that the terms of maximal degree in products of the form $T_{p_\lambda} T_{p_\tau w_0}$ are indexed by the set of admissible subsets J associated to a fix reduced expression of p_λ .

Then, according to the results of Lenart and Postnikov [9], we will obtain the desired equality.

The paper is organised as follows. In Section 2, we introduce all the needed material on (extended) affine Weyl groups. In Section 3, we present Kazhdan-Lusztig theory for affine Hecke algebras with unequal parameters and we describe the center of this algebra. In Section 4 and Section 5, we prove (1) and (2) above. Finally, we prove the desired equality using the theory of admissible subsets in Section 6. In Section 7, following [9], we describe the connection between admissible subsets and Lakshmibai-Seshadri paths.

2 Affine Weyl groups

Let V be an Euclidean space with scalar product (\cdot, \cdot) . We denote by V^* the dual of V and by $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbf{R}$ the canonical pairing. Let Φ be a root system and let Φ^\vee be the dual root system. If $\alpha \in \Phi$ then $\alpha^\vee \in \Phi^\vee$ is defined by $\langle x, \alpha^\vee \rangle = 2(x, \alpha)/(\alpha, \alpha)$. We fix a set of positive roots Φ^+ and a simple system Δ such that $\Delta \subset \Phi^+$.

2.1 Geometric presentation of an affine Weyl group

We denote by $H_{\alpha, n}$ the hyperplane defined by the equation $\langle x, \alpha^\vee \rangle = n$ and by \mathcal{F} the collection of all such hyperplanes. We will say that a hyperplane H is of direction $\alpha \in \Phi^+$ if there exists a pair $(\alpha, n) \in \Phi^+ \times \mathbf{Z}$ such that $H = H_{\alpha, n}$. We then write $\overline{H} = \alpha$.

Let Ω be the group generated by the set of orthogonal reflections $\sigma_{\alpha, n}$ with respect to $H_{\alpha, n}$ where $\alpha \in \Phi$ and $n \in \mathbf{Z}$. The group Ω is an affine Weyl group of type Φ^\vee . It is well known that Ω is generated by the set $\{\sigma_{\alpha, 0} \mid \alpha \in \Delta\} \cup \{\sigma_{\tilde{\alpha}, 1}\}$ where $\tilde{\alpha}$ is such that $\tilde{\alpha}^\vee$ is the highest root of Φ^\vee . To simplify notation, we will simply write σ_α for $\sigma_{\alpha, 0}$. Let Ω_0 be the stabiliser of 0 in Ω : clearly $\Omega_0 = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. Let σ_{Ω_0} be the longest element of Ω_0 .

The set of alcoves, denoted $\text{Alc}(\mathcal{F})$, is the set of connected components of $V \setminus \mathcal{F}$. The fundamental alcove A_0 is defined by

$$\begin{aligned} A_0 &= \{\lambda \in V \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \forall \alpha \in \Phi^+\} \\ &= \{\lambda \in V \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \forall \alpha \in \Delta\}. \end{aligned}$$

The group Ω acts simply transitively on the set of alcoves. We set $A_0^- = A_0 \sigma_{\Omega_0}$ and we have $A_0^- = A_0 \sigma_0$.

The group Ω acts on the set of faces (codimension 1 facet) of alcoves and we denote by S the set of orbits. Then to each $s \in S$, one can associate an involution on the set of $\text{Alc}(\mathcal{F})$: for all $A \in \text{Alc}(\mathcal{F})$, sA is the unique alcove which shares with A a face of type s . The set of all such involutions generate a Coxeter group W_a which is isomorphic to Ω . Both groups W_a and Ω acts transitively on the set of alcoves. We will write the action of W_a on the left and the action of Ω on the right. It is a well known fact that these two actions commute. For all $s \in S$ and all $A \in \text{Alc}(\mathcal{F})$, there exists a unique pair $(\alpha, n) \in \Phi^+ \times \mathbf{Z}$ such that the hyperplane $H_{\alpha, n}$ separate the alcoves A and sA and we have $sA = A \sigma_{\alpha, n}$.

Given two alcoves $A, B \in \text{Alc}(\mathcal{F})$, we set

$$H(A, B) = \{H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B\} \quad \text{and} \quad \overline{H(A, B)} = \{\overline{H} \mid H \in H(A, B)\}.$$

Let $A \in \text{Alc}(\mathcal{F})$, $\beta \in \Phi$ and $n \in \mathbf{Z}$. We write $n < A[\beta]$ if and only if for all $x \in A$ we have $n < \langle x, \beta^\vee \rangle$. Similarly, we write $n > A[\beta]$ if and only if for all $x \in A$ we have $n > \langle x, \beta^\vee \rangle$. For all alcoves A and all roots $\beta \in \Phi^+$, there exists a unique $n \in \mathbf{Z}$ such that $n < A[\beta] < n + 1$.

The following proposition gather some well known facts about the two actions introduced above.

Proposition 2.1. *Let $w \in W_a$. We have*

- 1) $\ell(w) = |H(A_0, wA_0)|$
- 2) *Let $s \in S$ and let $H_{\alpha, n}$ be the unique hyperplane separating wA_0 and swA_0 . We have $sw < w$ if and only if we are in one of the following situation:*
 - (a) $wA_0 \in H_{\alpha, n}^+$, $swA_0 \in H_{\alpha, n}^-$ and $n > 0$,
 - (b) $wA_0 \in H_{\alpha, n}^-$, $swA_0 \in H_{\alpha, n}^+$ and $n \leq 0$.
- 3) *We have $\overline{H(A_0, A_0 \sigma)} = \overline{H(A_0^-, A_0^- \sigma)} = \{\gamma \in \Phi^+ \mid \gamma \sigma^{-1} \in \Phi^-\}$.*

2.2 Weight functions and special points

Let L be a positive weight function on W_a , that is a function $L : W_a \rightarrow \mathbf{N}$ such that $L(ww') = L(w) + L(w')$ whenever $\ell(ww') = \ell(w) + \ell(w')$. To determine a weight function, it is enough to give its values on the conjugacy classes of generators of S . From now on, we fix such a positive weight function

L on W_a . Note that this also defines a weight function on Ω simply by setting $L(\sigma) = L(w)$ whenever $wA_0 = A_0\sigma$.

Let H be an hyperplane in \mathcal{F} . We say that H is of weight $L(r)$ if it contains a face of type $r \in S$. This is well-defined since if H contains a face of type r and r' then r and r' are conjugate and $L(r) = L(r')$ [2, Lemma 2.1]. We denote the weight of an hyperplane H by $L(H)$. If $\beta \in \Phi$, we set $L(\beta) = \max_{\overline{H}=\beta} L(H)$. For any $\lambda \in V$ we set

$$L(\lambda) = \sum_{H, \lambda \in H} L(H).$$

Let $\nu = \max_{\lambda \in V} L(\lambda)$. We call λ an L -weight if $L(\lambda) = \nu$ and we denote by P the set L -weights. Further we denote by P^+ the set of dominant L -weights that is $P^+ := \{\lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$ and by P^- the set of antidominant weights, that is $P^- = -P^+$. For $\lambda \in P$ and $\beta \in \Phi^+$ we set $\lambda_\beta := \langle \lambda, \beta^\vee \rangle$. For each root α , we set

$$b_\alpha = \begin{cases} 1 & \text{if } L_{H_{\alpha,0}} = L_{H_{\alpha,1}}, \\ 2 & \text{otherwise.} \end{cases}$$

Remark 2.2. The only case where there can be parallel hyperplanes with different weights and therefore some b_α equal to 2 is when W is of type C and when the extremal generator in the Dynkin diagram have same weights. We refer to [2, Lemma 2.2] for details on this case.

The action of the longest element σ_{Ω_0} on the set of weights is an involution, we set $\lambda^* = \lambda\sigma_{\Omega_0}$. Note that we do not have in general $\lambda\sigma_{\Omega_0} = -\lambda$ but if $\lambda \in P^+$ then $\lambda^* \in P^-$. If $\sigma = \sigma_{\beta_1} \dots \sigma_{\beta_n}$ where $\beta_i \in \Phi$ we set $\sigma^* = \sigma_{\Omega_0} \sigma \sigma_{\Omega_0} = \sigma_{\beta_1^*} \dots \sigma_{\beta_n^*}$. Let $v \in W_0$ and $\sigma_v \in \Omega_0$ be such that $vA_0 = A_0\sigma_v$. With our notation we have $vA_0^- = A_0^- \sigma_v^*$.

For $\lambda \in P$, we denote by W_λ the stabiliser in W_a of the set of alcoves that contain λ in their closure and by Ω_λ the stabiliser of λ in Ω . This notation is coherent since for all $\lambda \in P$ we have $W_\lambda \simeq \Omega_\lambda \simeq \Omega_0$ where $\Omega_0 = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$.

2.3 Extended affine Weyl groups

The extended affine Weyl group is defined by $\Omega_e = \Omega_0 \ltimes P$; it acts naturally on (the right) of V and on $\text{Alc}(\mathcal{F})$ but the action is no longer faithful. If we denote by Π the stabiliser of A_0 in Ω_e then we have $\Omega_e = \Pi \ltimes \Omega$. Note that Π permutes the weights that belongs to the closure of A_0 . Further the group Π is isomorphic to P/Q , hence abelian, and its action on Ω is given by an automorphism of the Dynkin diagram; see Planches I–IX in [1].

An extended alcove is a pair (A, μ) where $A \in \text{Alc}(\mathcal{F})$ and μ is a vertex of A which lies in P . We denote by $\text{Alc}_e(\mathcal{F})$ the set of extended alcoves. Then, the group Ω_e acts naturally on $\text{Alc}_e(\mathcal{F})$ and the action is faithful and transitive.

We set $W_e \simeq \Pi \ltimes W_a$. We define a left action of W_e on $\text{Alc}_e(\mathcal{F})$ as follows. Let $(A, \mu) \in \text{Alc}_e(\mathcal{F})$ and let $(a, s) \in \Pi \times S$. Let f_s be the face of type s of $A_0 \in \text{Alc}(\mathcal{F})$, by which we mean that $sA_0 = A_0\sigma_{f_s}$ where σ_{f_s} is the reflection with respect to the hyperplane that contains f_s . Let $\sigma_{A,\mu} \in \Omega_e$ be such that $(A_0, 0)\sigma_{A,\mu} = (A, \mu)$. We set

$$* s(A, \mu) = (A\sigma, \mu\sigma) \text{ where } \sigma \text{ is the reflection that contains the face } f_s\sigma_{A,\mu}.$$

$$* a(A, \mu) = (A, \mu a^{\sigma_A}) \text{ where } a^{\sigma_A} = \sigma_{A,\mu}^{-1} a \sigma_{A,\mu}.$$

It can be shown that the action of Ω_e and the action of W_e commute.

To simplify notation we will simply write $A_0 \in \text{Alc}_e(\mathcal{F})$ for the alcove $(A_0, 0)$. Similarly, we write A_λ and A_λ^- for the alcoves (A_λ, λ) and (A_λ^-, λ) . Let $p_\lambda \in W_e$ and p_λ^- be such that $p_\lambda A_0 = A_\lambda$ and $p_\lambda^- A_0 = A_\lambda^-$. Finally for all $w \in W_e$ we will denote by σ_w the unique element of Ω_e such that $w(A_0, 0) = (A_0, 0)\sigma_w$.

All the notion and notation for alcoves can be extended to $\text{Alc}_e(\mathcal{F})$. We just omit the part with the weight when needed. For instance if $A' = (A, \lambda) \in \text{Alc}_e(\mathcal{F})$, we write $A'[\gamma] < 0$ to mean $A[\gamma] < 0$. The length function, the weight function, the Bruhat order all naturally extend to W_e and Ω_e by setting $\ell(aw) = \ell(w)$, $L(aw) = L(w)$ and $aw < a'w'$ if and only if $a = a'$ and $w < w'$ where $a, a' \in \Pi$ and $w, w' \in W_a$.

2.4 Quarter of vertex λ

The quarters of vertex λ are the connected components of

$$V \setminus \bigcup_{H \in \mathcal{F}, \lambda \in H} H.$$

Given $\lambda \in P$ and $v \in W_0$, we let $\mathcal{C}_{\lambda,v}^+$ and $\mathcal{C}_{\lambda,v}^-$ the quarter of vertex λ which contains vA_λ and vA_λ^- respectively. When $v = 1$, we will omit the 1 in the notation $\mathcal{C}_{\lambda,1}$. The fundamental Weyl chamber is

$$\mathcal{C}_0^+ := \{x \in V \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi^+\}.$$

Let X_0 be the set of distinguished left coset representatives of W_0 in W_e . Any element w of W_e can be uniquely written under the form $w = xv$ where $v \in W_0$ and $x \in X_0$. In view of Proposition 2.1, we have

$$x \in X_0 \iff xA_0 \in \mathcal{C}_0^+ \iff xA_0[\alpha] > 0 \text{ for all } \alpha \in \Delta.$$

For all $x \in X_0$, $\lambda \in P$ and $v \in W_0$ we have $xvA_\lambda \in \mathcal{C}_{\lambda,v}^+$.

Let \mathcal{C} be a quarter of vertex $\lambda \in P$ and fix $\gamma \in \Phi^+$. We have either

$$\{\langle x, \gamma^\vee \rangle \mid x \in \mathcal{C}\} =]\lambda_\gamma, +\infty[\quad \text{or} \quad \{\langle x, \gamma^\vee \rangle \mid x \in \mathcal{C}\} =]-\infty, \lambda_\gamma[.$$

In the first case we say that \mathcal{C} is oriented toward $+\infty$ in the direction γ . In the second case we say that \mathcal{C} is oriented toward $-\infty$ in the direction γ .

Lemma 2.3. *Let $\lambda \in P$ and $v \in W_0$. We have for all $\gamma \in \Phi^+$:*

$$\mathcal{C}_{\lambda,v}^+[\gamma] = \begin{cases} +\infty & \text{if } \gamma \cdot \sigma_v^{-1} \in \Phi^+ \\ -\infty & \text{if } \gamma \cdot \sigma_v^{-1} \in \Phi^- \end{cases}$$

Proof. Let $x \in \mathcal{C}_{\lambda,v}^+$. There exists $x_0 \in \mathcal{C}_{0,v}^+$ and $y_0 \in \mathcal{C}_0^+$ such that $x = x_0 + \lambda$ and $x_0 = y_0\sigma_v$. We have

$$\langle x, \gamma^\vee \rangle = \langle x_0 + \lambda, \gamma^\vee \rangle = \langle y_0\sigma_v, \gamma^\vee \rangle + \lambda_\gamma = \lambda_\gamma + \langle y_0, (\gamma\sigma_v^{-1})^\vee \rangle$$

and the result follows by definition of \mathcal{C}_0^+ . □

Similarly, we have

$$\mathcal{C}_{\lambda,v}^-[\gamma] = \begin{cases} -\infty & \text{if } \gamma \cdot \sigma_v^{*-1} \in \Phi^+ \\ +\infty & \text{if } \gamma \cdot \sigma_v^{*-1} \in \Phi^- \end{cases}.$$

3 Affine Hecke algebras with unequal parameters

3.1 Affine Hecke algebra

Let $\mathcal{A} = \mathbf{C}[\mathbf{q}, \mathbf{q}^{-1}]$ where \mathbf{q} is an indeterminate. The Iwahori-Hecke algebra \mathcal{H} associated to W_e is the free \mathcal{A} -module with basis $\{T_w \mid w \in W_e\}$ and relation given by

$$T_u T_v = T_{uv} \text{ whenever } \ell(uv) = \ell(u) + \ell(v) \quad \text{and} \quad (T_s - \mathbf{q}^{L(s)})(T_s + \mathbf{q}^{-L(s)}) = 0 \text{ if } s \in S.$$

From this relation, we easily find that for all $s \in S$ and all $w \in W$, we have

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + \xi_s T_w, & \text{if } \ell(sw) < \ell(w) \end{cases} \quad \text{where } \xi_s = \mathbf{q}^{L(s)} - \mathbf{q}^{-L(s)}.$$

The basis $(T_w)_{w \in W_e}$ is called the standard basis. We write $f_{x,y,z}$ for the structure constants associated to the standard basis:

$$T_x T_y = \sum_{z \in W_e} f_{x,y,z} T_z.$$

The elements $f_{x,y,z}$ are polynomials in $\{\xi_s \mid s \in S\}$ with positive coefficients. The degree of $f_{x,y,z}$ will be denoted $\deg(f_{x,y,z})$ and is the highest power of \mathbf{q} that appear in $f_{x,y,z}$.

3.2 Multiplication of the standard basis

In this section, we present a result of [4] on a bound on the degree of the polynomials $f_{x,y,z}$. Recall that for $A, B \in \text{Alc}_e(\mathcal{F})$, we have set

$$H(A, B) = \{H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B\}.$$

Then for $x, y \in W_e$ we set

$$H_{x,y} = H(A_0, yA_0) \cap H(xyA_0, yA_0).$$

For $\alpha \in \Phi^+$ we set

$$c_{x,y}(\alpha) := \max_{\substack{H \in H_{x,y} \\ \overline{H} = \alpha}} L_H.$$

Then according to [4, Theorem 2.4] we have:

Theorem 3.1. *The degrees of the polynomials $f_{x,y,z}$ are bounded by $\sum_{\alpha \in \overline{H_{x,y}}} c_{x,y}(\alpha)$.*

We obtain the following corollary [5, Proposition 5.3] which will be crucial in the following section.

Corollary 3.2. *Let $y \in W_e$ and $x \in X_0$. Let $(y_0, y_r) \in W_0 \times X_0^{-1}$ be such that $y = y_0 y_r$. Then*

$$\deg(f_{x,y,z}) \leq L(w_0) - L(y_0).$$

3.3 Kazhdan-Lusztig basis

Let $\bar{\cdot}$ the ring involution of \mathcal{A} which takes \mathbf{q} to \mathbf{q}^{-1} . This involution can be extended to a ring involution of \mathcal{H} via the formula

$$\overline{\sum_{w \in W_e} a_w T_w} = \sum_{w \in W_e} \bar{a}_w T_w^{-1} \quad (a_w \in \mathcal{A}).$$

We set

$$\mathcal{A}_{<0} = \mathbf{q}^{-1} \mathbf{Z}[\mathbf{q}^{-1}] \quad \text{and} \quad \mathcal{H}_{<0} = \bigoplus_{w \in W_e} \mathcal{A}_{<0} T_w.$$

We define $\mathcal{A}_{\leq 0}$ and $\mathcal{H}_{\leq 0}$ in a similar manner. For each $w \in W_e$ there exists a unique element $C_w \in \mathcal{H}$ (see [11, Theorem 5.2]) such that (1) $\overline{C_w} = C_w$ and (2) $C_w \equiv T_w \pmod{\mathcal{H}_{<0}}$. For any $w \in W_e$ we set

$$C_w = T_w + \sum_{y \in W_e} P_{y,w} T_y \quad \text{where } P_{y,w} \in \mathcal{A}_{<0}.$$

The coefficients $P_{y,w}$ are called as the Kazhdan-Lusztig polynomials. It is well known ([11, §5.3]) that $P_{y,w} = 0$ whenever $y \not\leq w$. It follows that $(C_w)_{w \in W_e}$ forms an \mathcal{A} -basis of \mathcal{H} known as the Kazhdan-Lusztig basis.

Remark 3.3. Using Corollary 3.2 and the definition of the Kazhdan-Lusztig basis, one can show that $T_x C_{p_\lambda w_0} \in \mathcal{H}_{\leq 0}$ for all $\lambda \in P^+$.

3.4 The center of the affine Hecke algebra

In this section we follow the presentation of Nelsen and Ram [7] and we refer to it and the references therein for details and proofs.

We start by introducing another presentation of the affine Hecke algebra which is more convenient to describe its center $Z(\mathcal{H})$. For each $\lambda \in P$, we set $e^\lambda := T_{p_\mu} T_{p_\nu}^{-1}$ where $\mu, \nu \in P^+$ are such that $\mu - \nu = \lambda$. This can be shown to be independant of the choice of μ and ν . Then \mathcal{H} is generated by the sets $\{T_s \mid s \in S_0\}$ and $\{e^\lambda \mid \lambda \in P\}$ and we have the relations

$$e^\lambda e^\mu = e^{\lambda+\mu} = e^\mu e^\lambda \quad \text{and} \quad e^\lambda T_s = T_s e^{\lambda \sigma_{\alpha_i}} + \xi_s \frac{e^\lambda - e^{\lambda \sigma_{\alpha_i}}}{1 - e^{-\alpha_i}}$$

where s is such that $sA_0 = A_0 \sigma_{\alpha_i}$. Let $\mathcal{A}[P]$ be the subalgebra of \mathcal{H} generated by $\{e^\lambda \mid \lambda \in P\}$. The group Ω_0 acts naturally on $\mathcal{A}[P]$ via $e^\lambda \sigma = e^{\lambda \sigma}$.

Theorem 3.4. *The sets $\{e^\lambda T_w \mid \lambda \in P, w \in W_0\}$ and $\{T_w e^\lambda \mid w \in W_0, \lambda \in P\}$ are \mathcal{A} -basis of \mathcal{H} . The center $Z(\mathcal{H})$ of \mathcal{H} is*

$$\mathcal{A}[P]^{\Omega_0} = \{f \in \mathcal{A}[P] \mid f \cdot \sigma = f \text{ for all } \sigma \in \Omega_0\}.$$

Let $\mathcal{A}_-[P]^{\Omega_0} = \{f \in \mathcal{A}[P] \mid f \cdot \sigma = (-1)^{\ell(\sigma)} f \text{ for all } \sigma \in \Omega_0\}$ be the sets of antisymmetric functions and let

$$a_\mu := \sum_{\sigma \in \Omega_0} (-1)^{\ell(\sigma)} e^{\mu\sigma} \quad \text{and} \quad s_\mu = \frac{a_{\mu+\rho}}{a_\rho} \quad \text{where } \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

The elements s_μ which are called the Weyl characters are well-defined and the set $\{s_\mu \mid \mu \in P^+\}$ forms a basis of $\mathcal{A}[P]^{\Omega_0}$. Recall the definition of the coefficients $m_{\tau,\lambda}^\mu$ in the introduction.

Theorem 3.5. *We have $s_\tau C_{w_0} = C_{w_0} s_\tau = C_{p_\tau w_0}$ for all $\tau \in P^+$ and*

$$s_\tau C_{p_\lambda w_0} = s_\tau s_\lambda C_{w_0} = \sum_{\mu} m_{\tau,\lambda}^\mu C_{p_\mu w_0}.$$

4 Decomposition into the Kazhdan-Lusztig basis

The aim of this section is to prove Statement (1) in the introduction, that is, in order to determine the decomposition of $s_\tau C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis, it is enough to determine the terms of maximal degree in product of the form $T_{p_\tau} T_{p_\lambda w_0}$.

Let $x \in X_0$ (respectively $y \in X_0^{-1}$). There exists a unique family of polynomials $(p_{x',x})_{x' \in X_0}$ in $\mathcal{A}_{<0}$ (respectively $(p_{x',x}^r)_{x' \in X_0}$ in $\mathcal{A}_{<0}$) such that $p_{x',x} = 0$ whenever $x' \not\leq x$ (respectively $p_{y',y}^r = 0$) and

$$C_{xw_0} = T_x C_{w_0} + \sum_{x' \in X_0} p_{x',x} T_{x'} C_{w_0} \quad \text{and} \quad C_{w_0 y} = C_{w_0} T_y + \sum_{y' \in X_0^{-1}} p_{y',y}^r C_{w_0} T_{y'}.$$

The polynomials p are called relative Kazhdan-Lusztig polynomials [3]. In our case, it can be shown [12] that we have $p_{x,x'} = P_{xw_0, x'w_0}$ although it is sufficient for us to know that they belong to $\mathcal{A}_{<0}$. Finally, we set

$$\mathbf{P}(x) = \sum_{x' \in X_0} p_{x',x} T_{x'} \quad \text{and} \quad \mathbf{P}_R(y) = \sum_{y' \in X_0^{-1}} p_{y',y}^r T_{y'}.$$

so that $\mathbf{P}(x) C_{w_0} = C_{xw_0}$ and $C_{w_0} \mathbf{P}_R(y) = C_{w_0 y}$. When $\tau \in P^+$, we know that $p_\tau \in X_0$. To lighten notation, we will write $\mathbf{P}(\tau)$ instead of $\mathbf{P}(p_\tau)$. For all $\tau, \lambda \in P^+$ we have

$$s_\tau C_{w_0} = \mathbf{P}(\tau) C_{w_0} \quad \text{and} \quad s_\tau C_{p_\lambda w_0} = s_\tau C_{w_0} s_\lambda = \mathbf{P}(\tau) C_{w_0} s_\lambda = \mathbf{P}(\tau) C_{p_\lambda w_0}.$$

Let $\tau, \lambda \in P^+$. Using Remark 3.3 we get

$$\begin{aligned} \mathbf{P}(\tau) C_{p_\lambda w_0} &= T_{p_\tau} C_{p_\lambda w_0} + \sum_{x < p_\tau, x \in X_0} \underbrace{p_{x,p_\tau} T_x C_{p_\lambda w_0}}_{\in \mathcal{H}_{<0}} \\ &\equiv T_{p_\tau} C_{p_\lambda w_0} \pmod{\mathcal{H}_{<0}}. \end{aligned}$$

Next

$$T_{p_\tau} C_{p_\lambda w_0} = T_{p_\tau} T_{p_\lambda w_0} + \sum_{y < p_\lambda w_0} P_{y,p_\lambda w_0} T_{p_\tau} T_y.$$

Let $y \in W_e$ and $(y_0, y_r) \in W_0 \times X_0^{-1}$ be such $y = y_0 y_r$. On the one hand, we have

$$P_{y,p_\lambda w_0} = P_{y_0 y_r, w_0 p_\lambda} = \mathbf{q}^{L(y_0) - L(w_0)} P_{w_0 y_r, w_0 p_\lambda}.$$

On the other hand the maximal degree that can appear in $T_{p_\tau} T_y$ is $L(w_0) - L(y_0)$. Therefore if $w_0 y_r < w_0 p_\lambda$ we get that $P_{y,p_\lambda w_0} T_{p_\tau} T_y \in \mathcal{H}_{<0}$ and

$$\begin{aligned} T_{p_\tau} C_{p_\lambda w_0} &\equiv T_{p_\tau} T_{p_\lambda w_0} + \sum_{y_0 \in W_0} \mathbf{q}^{L(y_0) - L(w_0)} T_{p_\tau} T_{y_0 p_\lambda} \pmod{\mathcal{H}_{<0}} \\ &\equiv T_{p_\tau} T_{w_0 p_\lambda} + \sum_{v \in W_0} \mathbf{q}^{-L(v)} T_{p_\tau} T_{(vw_0) p_\lambda} \pmod{\mathcal{H}_{<0}}. \end{aligned}$$

Finally

$$\mathbf{P}(\tau) C_{p_\lambda w_0} \equiv T_{p_{\tau+\lambda} w_0} + \sum_{v \in W_0} \underbrace{\mathbf{q}^{-L(v)} T_{p_\tau} T_{(vw_0) p_\lambda}}_{\in \mathcal{H}_{<0}} \pmod{\mathcal{H}_{<0}}.$$

The element $\mathbf{P}(\tau)C_{w_0 p_\lambda^*}$ are stable under the $-$ -involution, thus to determine the decomposition of $\mathbf{P}(\tau)C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis, we need to determine which products $T_{p_\tau}T_{(vw_0)p_\lambda^*}$ can actually give rise to a coefficient of degree $L(v)$. To simplify notation we now take $\lambda \in P^-$.

Let $x, y \in W_e$ and fix a reduced expression $as_n \dots s_1$ of x where $a \in \Pi$ and $s_i \in S$ for all i . Let $J = \{i_1, \dots, i_p\}$ be a subset of $\{1, \dots, n\}$. For all $1 \leq \ell \leq k < n$, we set

$$x^J_{[k, \ell]} = \prod_{r=\ell, r \notin J}^k s_r \quad \text{and} \quad x^J_{[n, \ell]} = a \prod_{r=\ell, r \notin J}^n s_r.$$

We denote by $\mathcal{J}_{x, y}$ the set of all subsets $\{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$ such that $1 \leq i_1 < \dots < i_p \leq k$ and

$$s_{i_t} x^J_{[i_t-1, 1]} < x^J_{[i_t-1, 1]} \text{ for all } t \in \{1, \dots, p\}.$$

It should be noted that the set $\mathcal{J}_{x, y}$ depends on the reduced expression of x . For $J = \{i_1, \dots, i_p\}$ in $\mathcal{J}_{x, y}$, we set $\xi_J = \prod_{k=1}^p \xi_{s_{i_k}}$. Then we have [2, Proof of Proposition 5.1]

$$T_x T_y = \sum_{J \in \mathcal{J}_{x, y}} \xi_J T_{x^J_{[n, 1]} y}.$$

We fix a reduced expression $as_n \dots s_1$ of p_τ . Then

$$T_{p_\tau} T_{vw_0 p_\lambda} = T_{p_\tau} T_{vp_\lambda^-} = \sum_{J \in \mathcal{J}_{p_\tau, vp_\lambda^-}} \xi_J T_{p_\tau^J_{[n, 1]} vp_\lambda^-}.$$

If we set (keeping in mind that a reduced expression for p_τ is fixed)

$$\mathcal{J}_{\tau, v\lambda} := \mathcal{J}_{p_\tau, vp_\lambda^-} \quad \text{and} \quad \mathcal{J}_{\tau, v\lambda}^{\max} := \{J \in \mathcal{J}_{p_\tau, vp_\lambda^-} \mid \xi_J = L(v)\}$$

then, we have

$$\mathbf{P}(\tau)C_{w_0 p_\lambda^-} = \sum_{v \in W_0} \sum_{J \in \mathcal{J}_{\tau, v\lambda}^{\max}} C_{p_\tau^J_{[n, 1]} vp_\lambda^-}$$

we now need to determine the sets $\mathcal{J}_{\tau, v\lambda}^{\max}$ for all $v \in W_0$ and all $\lambda \in P^-$.

5 Description of $\mathcal{J}_{\tau, v\lambda}^{\max}$ in terms of admissible subsets

Once and for all in this section, we fix a reduced expression $p_\tau = as_n \dots s_1$ where $a \in \Pi$ and $s_i \in S$. Let $(\beta_1, \dots, \beta_n) \in (\Phi^+)^n$ be such that

$$s_i \dots s_1 A_0 = A_0 \sigma_{\beta_1, N_1} \dots \sigma_{\beta_i, N_i} \text{ for all } i \in \{1, \dots, n\}.$$

Recall that for $\beta \in \Phi^+$, we have set $\beta^* := \beta \sigma_{\Omega_0} \in \Phi^-$. We have

$$p_\tau A_0^- = p_\tau A_0 \sigma_{\Omega_0} = A_0 \sigma_{\beta_1, N_1} \dots \sigma_{\beta_n, N_n} \sigma_{\Omega_0} = A_0^- \sigma_{\beta_1^*, N_1} \dots \sigma_{\beta_n^*, N_n}.$$

Following [9], we now introduce the concept of admissible subsets. Recall that a chain $(\sigma_0, \sigma_1, \dots, \sigma_p) \in \Omega_0^p$ is an increasing saturated chain in the Bruhat order if and only if

$$\sigma_{i-1} < \sigma_i \quad \text{and} \quad \ell(\sigma_i) = \ell(\sigma_{i-1}) + 1 \text{ for all } i \in \{1, \dots, p\}.$$

Definition 5.1. A subset $J = \{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$ will be called an admissible subset if

$$e < \sigma_{\beta_{i_p}} < \sigma_{\beta_{i_{p-1}}} \sigma_{\beta_{i_p}} < \dots < \sigma_{\beta_1} \dots \sigma_{\beta_{i_{p-1}}} \sigma_{\beta_{i_p}}$$

is a saturated chain in the Bruhat order on Ω_0 .

Conjugating by σ_{Ω_0} , we see that $J = \{i_1, \dots, i_p\}$ is admissible if and only if

$$e < \sigma_{\beta_{i_p}^*} < \sigma_{\beta_{i_{p-1}}^*} \sigma_{\beta_{i_p}^*} < \dots < \sigma_{\beta_1^*} \dots \sigma_{\beta_{i_{p-1}}^*} \sigma_{\beta_{i_p}^*}$$

is a saturated chain in the Bruhat order on Ω_0 .

Remark 5.2. Our definition of admissible subset is slightly different than the one in [9] where they multiply on the right in the definition. This is because, in our work, reduced expressions of p_τ naturally appear whereas the authors in [9] work with reduced expressions of $p_{-\tau}$. The relationship between those two definitions will be made explicit in the next section.

Definition 5.3. Let $J = \{i_1, \dots, i_p\}$ be an admissible subset. Let $\sigma_J = \sigma_{\beta_1} \dots \sigma_{\beta_{i_{p-1}}} \sigma_{\beta_{i_p}}$ and $v_J \in W_0$ be such that $v_J A_0 = A_0 \sigma_J$. We say that J is

- 1) μ -dominant for $\mu \in P^+$ if the alcoves $p_\tau^J[k, 1]v_J A_\mu \in \mathcal{C}_0^+$ for all $k \in \{1, \dots, n\}$,
- 2) maximal if $L(s_{i_\ell}) = L(\beta_{i_\ell})$ for all $1 \leq \ell \leq p$.

If $\mu \in P^-$, we will say that J is λ -antidominant if $p_\tau^J[k, 1]v_J A_\mu^- \in \mathcal{C}_0^-$ for all $k \in \{1, \dots, n\}$. We can easily see that J is μ -antidominant if and only if J is μ^* -dominant.

We have $L(\sigma_J) = L(v_J) = \sum_{k=1}^p L(\beta_{i_k})$. In particular, we see that if $J \in \mathfrak{J}_{\tau, v_J \lambda}^{\max}$, then J must be maximal in order to have $\deg(\xi_J) = L(v_J)$.

Let $\beta \in \Phi^+$ and $\sigma \in \Omega_0$. We have [6, §5.7] $\sigma < \sigma_\beta \sigma$ if and only if $\beta \sigma \in \Phi^+$. If $J = \{i_1, \dots, i_p\}$ is an admissible subset, then, since $\sigma_{\beta_{i_{\ell+1}}} \dots \sigma_{\beta_{i_p}} < \sigma_{\beta_{i_\ell}} \sigma_{\beta_{i_{\ell+1}}} \dots \sigma_{\beta_{i_p}}$, we have $\beta_{i_\ell} \sigma_{\beta_{i_{\ell+1}}} \dots \sigma_{\beta_{i_p}} \in \Phi^+$. Similarly, since β_i^* is a negative root, we have

$$\sigma_{\beta_{i_{\ell+1}}}^* \dots \sigma_{\beta_{i_p}}^* < \sigma_{\beta_{i_\ell}}^* \sigma_{\beta_{i_{\ell+1}}}^* \dots \sigma_{\beta_{i_p}}^* \implies (-\beta_{i_\ell}^*) \sigma_{\beta_{i_{\ell+1}}}^* \dots \sigma_{\beta_{i_p}}^* \in \Phi^+ \implies (\beta_{i_\ell}^*) \sigma_{\beta_{i_\ell}}^* \dots \sigma_{\beta_{i_p}}^* \in \Phi^+.$$

Note that if $\{i_1, \dots, i_p\}$ is an admissible subset then so is $\{i_\ell, \dots, i_p\}$ for all $\ell \leq p$: we'll denote this subset by $J_{\ell-1}$ so that $J_0 = J$ and $J_p = \emptyset$. Then according to Definition 5.3 we have

$$\sigma_{J_\ell} = \sigma_{\beta_{i_{\ell+1}}} \dots \sigma_{\beta_{i_p}} \quad \text{and} \quad v_{J_\ell} A_0 = A_0 \sigma_{J_\ell}.$$

We are now ready to state the main result of this section. Recall the notation introduced at the end of the previous section.

Theorem 5.4. Let $\lambda \in P^-$ and $v \in W_0$.

- 1) If J is admissible, λ -antidominant and maximal then $J \in \mathfrak{J}_{\tau, v_J \lambda}^{\max}$.
- 2) If $J \in \mathfrak{J}_{\tau, v\lambda}^{\max}$ then $v = v_J$, J is admissible, λ -antidominant and maximal.

The rest of this section is devoted to the proof of this theorem.

Proposition 5.5. Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \dots, n\}$.

- 1) The unique hyperplane separating the alcoves $p_\tau[k-1, 1]v A_\lambda$ and $p_\tau[k, 1]v A_\lambda$ is $H_k := H_{\beta_k \sigma_v, N_k + \langle \lambda, \beta_k \sigma_v^\vee \rangle}$.
- 2) We have $p_\tau[k-1, 1]v A_\lambda = p_\tau[k, 1]v' A_{\lambda'}$ where $\sigma_{v'} = \sigma_{\beta_k} \sigma_v$ and $\lambda' = \lambda \sigma_{H_k} = \lambda + N_k \beta_k \sigma_v$.

Proof. The (unique) hyperplane separating $p_\tau[k-1, 1]A_0$ and $p_\tau[k, 1]A_0$ is H_{β_k, N_k} . It follows that the hyperplane separating $p_\tau[k-1, 1]v A_0$ and $p_\tau[k, 1]v A_0$ is $H_{\beta_k, N_k} \sigma_v = H_{\beta_k \sigma_v, N_k}$. Finally, the hyperplane separating $p_\tau[k-1, 1]v A_\lambda$ and $p_\tau[k, 1]v A_\lambda$ is $H_{\beta_k \sigma_v, N_k + \langle \lambda, \beta_k \sigma_v^\vee \rangle}$. Next we have

$$\begin{aligned} p_\tau[k-1, 1]v A_\lambda &= p_\tau[k, 1]v A_\lambda \sigma_{H_k} \\ &= p_\tau[k, 1]v A_{\lambda'} \sigma_{\beta_k \sigma_v, \langle \lambda', \beta_k \sigma_v^\vee \rangle} \\ &= p_\tau[k, 1]v A_0 \sigma_{\beta_k \sigma_v} t_{\lambda'} \\ &= p_\tau[k, 1]A_0 \sigma_v \sigma_{\beta_k \sigma_v} t_{\lambda'} \\ &= p_\tau[k, 1]A_0 \sigma_{\beta_k} \sigma_v t_{\lambda'} \\ &= p_\tau[k, 1]v' A_{\lambda'} \end{aligned}$$

as required. \square

Given any triplet (λ, v, J) such that $(\lambda, v) \in P \times W_0$ and $J = \{i_1, \dots, i_p\} \in \mathfrak{J}_{\tau, v\lambda}$, we associate sequences $\underline{v} = (v_0, \dots, v_p)$, $\underline{\gamma} = (\gamma_1, \dots, \gamma_p)$ and $\underline{\lambda} = (\lambda_0, \dots, \lambda_p)$ defined by the following relations for $\ell \in \{1, \dots, p\}$:

- $v_0 = v$, $\sigma_0 = \sigma_v$ and $v_\ell A_0 = A_0 \sigma_{\beta_{i_\ell}} \dots \sigma_{\beta_{i_1}} \sigma_v$;
- $\gamma_{i_\ell} = \beta_{i_\ell}^* \sigma_{v_{\ell-1}}^*$;
- $\lambda_0 = \lambda$ and $\lambda_\ell = \lambda_{\ell-1} + N_{i_\ell} \gamma_{i_\ell} = \lambda_{\ell-1} \sigma_{\gamma_{i_\ell}, N_{i_1} + \langle \lambda_{\ell-1}, \gamma_{i_\ell}^\vee \rangle}$.

With these notation, we have for all $0 \leq \ell \leq p$:

$$p_\tau^J[i_\ell-1, 1]v A_\lambda^- = p_\tau[i_\ell-1, 1]v_{\ell-1} A_{\lambda_{\ell-1}}^- = p_\tau[i_\ell, 1]v_\ell A_{\lambda_\ell}^-.$$

If we choose $v = v_J$ then $v_\ell = v_{J_\ell}$, $\gamma_{i_\ell} = \beta_{i_\ell}^* \sigma_{\beta_{i_\ell}}^* \dots \sigma_{\beta_{i_p}}^*$ and

$$\sigma_{\beta_{i_\ell}}^* \dots \sigma_{\beta_{i_p}}^* = \sigma_{\gamma_{i_p}} \dots \sigma_{\gamma_{i_\ell}}.$$

We are now ready to prove the first part of Theorem 5.4.

Proposition 5.6. *Let $\lambda \in P^-$. If J is admissible, λ -antidominant and maximal then $J \in \mathfrak{J}_{\tau, vJ\lambda}^{\max}$.*

Proof. In order to prove that $J = \{i_1, \dots, i_p\} \in \mathfrak{J}_{\tau, vJ\lambda}^{\max}$, since J is maximal, we only need to show that

$$s_{i_\ell} p_\tau^J [i_\ell - 1, 1] v_J p_\lambda^- < p_\tau^J [i_\ell - 1, 1] v_J p_\lambda^- \text{ for all } 1 \leq \ell \leq p.$$

We have

- 1) $p_\tau^J [i_\ell - 1, 1] v_J p_\lambda^- = p_\tau [i_\ell - 1, 1] v_{J_{\ell-1}} p_{\lambda_{\ell-1}}^-$;
- 2) $(-\beta_{i_\ell}^*) \sigma_{\beta_{i_{\ell+1}}^*} \dots \sigma_{\beta_{i_p}^*} = \beta_{i_\ell}^* \sigma_{J_{\ell-1}} = \gamma_{i_\ell} \in \Phi^+$ (since J is admissible);
- 3) the hyperplane separating $p_\tau^J [i_\ell - 1, 1] v_J A_\lambda^-$ and $s_{i_\ell} p_\tau^J [i_\ell - 1, 1] v_J A_\lambda^-$ is equal to $H_{\gamma_{i_\ell}, m}$ where $m < 0$ (since J is λ -antidominant).

It follows from the second statement that $\gamma_{i_\ell} \sigma_{J_{\ell-1}}^{*-1} = \beta_{i_\ell}^* \in \Phi^-$. The quarter $\mathcal{C}_{\lambda_{\ell-1}, v_{J_{\ell-1}}}^-$ is therefore oriented towards $+\infty$ in the direction γ_{i_ℓ} . The result follows by Proposition 2.1 and statement 3). \square

We now focus on the second part. We start by proving some technical lemmas.

Lemma 5.7. *Let $\lambda \in P^-$ and $v \in W_0$. We have $\overline{H_{p_\tau, v p_\lambda^-}} \subset \{\delta \in \Phi^+ \mid \delta \sigma_v^{*-1} \in \Phi^-\}$.*

Proof. Let $\delta \in \Phi^+$ be such that $H_{\delta, N} \in H_{p_\tau, v p_\lambda^-} = H(A_0, v A_\lambda^-) \cap H(p_\tau v A_\lambda^-, v A_\lambda^-)$. First of all, since $\lambda \in P^-$, we have $v A_\lambda^- [\delta] < 1$. It follows that $v A_\lambda^- [\delta] < N \leq 0$. If the quarter $\mathcal{C}_{\lambda, v}^-$ is oriented towards $-\infty$ in the direction δ then $v A_\lambda^- [\delta] \geq p_\tau v A_\lambda^- [\delta]$ so that $v A_\lambda^- [\delta] > N$ which is impossible. This shows that the quarter $\mathcal{C}_{\lambda, v}^-$ is oriented towards $+\infty$ in the direction δ and thus $\delta \sigma_v^{*-1} \in \Phi^-$ as required. \square

We remark that if $v A_\lambda \notin \mathcal{C}_0^-$ then there exists a root $\delta \in \Phi^+$ such that $0 < v A_\lambda^- [\delta] < 1$ and $\delta \sigma_v^{*-1} \in \Phi^-$ which in turn implies that the inclusion $\overline{H_{p_\tau, v p_\lambda^-}} \subset \{\delta \in \Phi^+ \mid \delta \sigma_v^{*-1} \in \Phi^-\}$ is strict. Then $\mathfrak{J}_{\tau, v\lambda}^{\max}$ has to be empty by Theorem 3.1.

Lemma 5.8. *Let $\lambda \in P$ and $v \in W_0$. If $J = \{i_1, \dots, i_p\} \in \mathfrak{J}_{\tau, v\lambda}$ is such that $p_\tau [k_0, 1] v A_\lambda^- \in \mathcal{C}_0^-$ for some $k_0 \in \{0, \dots, i_1 - 1\}$ then $\overline{H_{p_\tau [n, i_1], p_\tau [i_1 - 1, 1] v p_\lambda^-}} \subset \{\delta \in \Phi^+ \mid \delta \sigma_v^{*-1} \in \Phi^-\}$.*

Proof. Let $\delta \in \Phi^+$ be such that $H_{\delta, N} \in H_{p_\tau [n, i_1], p_\tau [i_1 - 1, 1] v p_\lambda^-}$ where we recall that

$$H_{p_\tau [n, i_1], p_\tau [i_1 - 1, 1] v p_\lambda^-} = H(A_0, p_\tau [i_1 - 1, 1] v A_\lambda^-) \cap H(p_\tau v A_\lambda^-, p_\tau [i_1 - 1, 1] v A_\lambda^-).$$

Assume first that the $\mathcal{C}_{\lambda, v}$ is oriented towards $-\infty$. Then we must have

$$v A_\lambda^- [\delta] \geq \underbrace{p_\tau [k_0, 1] v A_\lambda^- [\delta]}_{< 0} \geq p_\tau [i_1 - 1, 1] v A_\lambda^- [\delta] \geq p_\tau v A_\lambda^- [\delta].$$

But $H_{\delta, N} \in H(A_0, p_\tau [i_1 - 1, 1] v A_\lambda^-)$ implies that $0 \geq N > p_\tau [i_1 - 1, 1] v A_\lambda^- [\delta]$. Therefore in this case we cannot have $H_{\delta, N} \in H(p_\tau v A_\lambda^-, p_\tau [i_1 - 1, 1] v A_\lambda^-)$. This shows that the quarter $\mathcal{C}_{\lambda, v}^-$ has to be oriented towards $+\infty$ in the direction δ and therefore we have $\delta \sigma_v^{*-1} \in \Phi^-$. \square

Lemma 5.9. *Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \dots, n\}$ such that*

$$s_k p_\tau [k-1, 1] v p_\lambda^- < p_\tau [k-1, 1] v p_\lambda^- \quad \text{and} \quad p_\tau [k-1, 1] v A_\lambda^- \in \mathcal{C}_0^-.$$

Then $\beta_k^ \sigma_v^* \in \Phi^+$.*

Proof. The unique hyperplane H that separates $p_\tau [k-1, 1] v A_\lambda^-$ and $s_k p_\tau [k-1, 1] v A_\lambda^-$ is

$$H := H_{\beta_k^* \sigma_v^*, N_k + \langle \lambda, \beta_k^* \sigma_v^{*\vee} \rangle}.$$

Since $p_\tau [k-1, 1] v A_\lambda^- \in \mathcal{C}_0^-$, we must have $p_\tau [k-1, 1] v A_\lambda^- \in H^-$ and $s_k p_\tau [k-1, 1] v A_\lambda^- \in H^+$, which means that the quarter $\mathcal{C}_{\lambda, v}^-$ has to be oriented toward $+\infty$ in the direction $|\beta_k^* \sigma_v^*| \in \Phi^+$. According to Lemma 2.3, since $\beta_k^* \sigma_v^* \sigma_v^{*-1} = \beta_k^* \in \Phi^-$, we must have $\beta_k^* \sigma_v^* \in \Phi^+$. \square

Proposition 5.10. *Let $\lambda \in P^-$, $v \in W_0$. If $J \in \mathfrak{J}_{\tau, v\lambda}^{\max}$ then J is λ -antidominant.*

Proof. Let $J = \{i_1, \dots, i_p\} \in \mathfrak{J}_{\tau, v\lambda}^{\max}$. We have already seen that if $vA_{\lambda}^- \notin \mathcal{C}_0^-$ then $\mathfrak{J}_{\tau, v\lambda}^{\max} = \emptyset$. Thus we must have $vA_{\lambda}^- \in \mathcal{C}_0^-$. Assume that there exists an index k_0 such that $p_{\tau}^J[k_0+1, 1]vA_{\lambda}^- \notin \mathcal{C}_0^-$ and choose k_0 to be minimal with this property. Let $0 \leq \ell \leq p$ be such that $i_{\ell+1} > k_0 + 1 > i_{\ell}$ where we set $i_0 = 0$ and $i_{p+1} = +\infty$. By minimality of k_0 we get $p_{\tau}^J[k', 1]vA_{\lambda}^- \in \mathcal{C}_0^-$ for all $k' \leq k_0$.

We construct the sequences \underline{v} and $\underline{\lambda}$ associated to the triplet (λ, v, J) . By construction and by hypothesis we have

$$s_{i_{\ell}} p_{\tau}[i_{\ell}-1, 1]v_{\ell-1} p_{\lambda_{\ell-1}}^- < p_{\tau}[i_{\ell}-1, 1]v_{\ell-1} p_{\lambda_{\ell-1}}^- \quad \text{and} \quad p_{\tau}[i_{\ell}-1, 1]v_{\ell-1} A_{\lambda_{\ell-1}}^- \in \mathcal{C}_0^-.$$

Therefore, by Lemma 5.9, we have $\beta_{i_{\ell}}^* \sigma_{v_{\ell-1}}^* \in \Phi^+$, in other words $(-\beta_{i_{\ell}}^*) \sigma_{v_{\ell}}^* \in \Phi^+$. This implies in particular that $\sigma_{v_{\ell}} = \sigma_{\beta_{i_{\ell}}^*} \sigma_{v_{\ell-1}} < \sigma_{v_{\ell-1}}$. Thus we have $\sigma_{v_0} > \sigma_{v_1} \dots > \sigma_{v_{\ell}}$ and $L(\sigma_{v_{\ell}}) = L(\sigma_v) - \sum_{k=1}^{\ell} L(\beta_{i_k}^*)$. Since $J \in \mathfrak{J}_{\tau, v\lambda}^{\max}$ we have

$$(\star) \quad L(\sigma_{v_{\ell}}) = L(\sigma_{v_0}) - \sum_{k=1}^{\ell} L(\beta_{i_k}^*) = \sum_{k=\ell+1}^p L(\beta_{i_k}^*) = \sum_{k=\ell+1}^p L(s_{i_k}).$$

In particular, if $\ell = p$, then we must have $v_p = 1$. We know that

$$p_{\tau}^J[k_0, 1]vA_{\lambda}^- = p_{\tau}[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^- \quad \text{and} \quad s_{k_0+1} p_{\tau}^J[k_0, 1]vA_{\lambda}^- = s_{k_0+1} p_{\tau}[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^-.$$

We denote by $H := H_{\delta_0, 0}$ the unique hyperplane separating $s_{k_0+1} p_{\tau}^J[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^-$ and $p_{\tau}^J[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^-$. Since $p_{\tau}^J[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^- \in \mathcal{C}_0^-$ and $s_{k_0+1} p_{\tau}^J[k_0, 1]v_{\ell} A_{\lambda_{\ell}}^- \notin \mathcal{C}_0^-$, the quarter $\mathcal{C}_{\lambda_{\ell}, v_{\ell}}$ has to be oriented toward $+\infty$ in the direction δ_0 . This implies $\delta_0 \in \{\gamma \in \Phi^+ \mid \gamma \sigma_{v_{\ell}}^{*-1} \in \Phi^-\}$. Further, we have $\ell < p$ since otherwise $v_{\ell} = 1$ and there is no such δ_0 .

The set $J_{\ell} = \{i_{\ell+1}, \dots, i_p\} \in \mathfrak{J}_{\tau, v_{\ell}\lambda_{\ell}}$ is non-empty and $p_{\tau}[k_0, 1]A_{\lambda_{\ell}}^- \in \mathcal{C}_0^-$ (where $k_0 < i_{\ell+1}$) thus by Lemma 5.8 we know that

$$\overline{H_{p_{\tau}[n, i_{\ell+1}], p_{\tau}[i_{\ell+1}-1, 1]v_{\ell} p_{\lambda_{\ell}}^-}} \subset \{\gamma \in \Phi^+ \mid \gamma \sigma_{v_{\ell}}^{*-1} \in \Phi^-\}.$$

By definition of δ_0 and since $i_{\ell+1} > k_0 + 1$ we have

$$\overline{H_{p_{\tau}[n, i_{\ell+1}], p_{\tau}[i_{\ell+1}-1, 1]v_{\ell} p_{\lambda_{\ell}}^-}} \subset \{\gamma \in \Phi^+ \mid \gamma \sigma_{v_{\ell}}^{*-1} \in \Phi^-\} \setminus \{\delta_0\}$$

and by Theorem 3.1 we obtain

$$\sum_{k=\ell+1}^p L(s_{i_k}) < \sum_{\gamma \in \Phi^+, \gamma \sigma_{v_{\ell}}^{*-1} \in \Phi^-} L(\gamma) = L(v_{\ell}).$$

This contradicts (\star) and therefore, there cannot be such an index k_0 . \square

Now that we know that if $J \in \mathfrak{J}_{\lambda, v}^{\max}$ then J is λ -antidominant, we can re-do the first part of the proof and show that we must have $v = v_J$ (indeed in the case where J is λ -antidominant there is no index k_0 and therefore $\ell = p$).

The proof of the second part of Theorem 5.4 follows from Proposition 5.10.

Corollary 5.11. *Let $\lambda \in P^-$ and $\tau \in P^+$. We have*

$$P(\tau)C_{w_0 p_{\lambda}} = \sum_J C_{p_{\tau}^J v_J w_0 p_{\lambda}}$$

where the sum is taken over all λ -antidominant maximal admissible subset J .

Remark 5.12. If we are working in the case of equal parameters, we can remove the condition of maximality since all parallel hyperplanes have same weights and therefore all admissible subsets are maximal.

6 Proof of the main result

Fix a reduced expression $as_n \dots s_1$ of p_τ and denote by H_{β_i, N_i} the corresponding hyperplanes as in the beginning of the previous section. As mentioned in Remark 5.2, the authors in [9] work with reduced expression of $p_{-\tau}$. Let $a^{-1}s'_n \dots s'_1$ be the reduced expression of $p_{-\tau}$ obtained by inverting the reduced expression above and moving a^{-1} to the left. Then, the hyperplane separating $s'_i \dots s'_1 A_0$ and $s'_{i-1} \dots s'_1 A_0$ is $H_{\beta'_i, N'_i}$ where $\beta'_i = \beta_{n-i+1}$ and $N'_i = N_{n-i+1} - \langle \tau, \beta_{n-i+1}^\vee \rangle$. The map $i \mapsto n - i + 1$ induces a bijection $J \mapsto J^\dagger$ between admissible subsets in the sense of Definition 5.1 and admissible subsets in the sense of Lenart and Postnikov [9, Definition 6.1]. To distinguish between those two definitions, we will say that a subset is LP-admissible if it is admissible in the sense of [9].

We start with the definition of a gallery.

Definition 6.1. A gallery is a sequence $\gamma = (\mu_0, B_0, F_1, B_1, \dots, B_\ell, \mu_{\ell+1})$ such that

- 1) B_0, \dots, B_ℓ are alcoves in $\text{Alc}(\mathcal{F})$;
- 2) for all $k \in \{1, \dots, \ell\}$, F_k is the common facet of B_k and B_{k-1} ;
- 3) μ_0 and $\mu_{\ell+1}$ are vertices of B_0 and B_ℓ respectively which are elements of P .

The weight of the gallery is $F_{\ell+1} - F_0$ and we denote it by $\mathbf{w}(\gamma)$.

Let $\gamma = (\mu_0, B_0, F_1, B_1, \dots, B_\ell, \mu_{\ell+1})$ be a gallery and let σ_k be the affine reflection with respect to the hyperplane which contains the facet F_k . The k -th tail flip operator f_k is defined by

$$f_k(\gamma) = (F_0, B_0, F_0, \dots, B_{k-1}, F_k = F'_k, \underset{B_k \sigma_k}{\parallel} B'_k, \underset{F_{k+1} \sigma_k}{\parallel} F'_{k+1}, \dots, \underset{B_\ell \sigma_k}{\parallel} B'_\ell, \underset{F_{\ell+1} \sigma_k}{\parallel} F'_{\ell+1}).$$

The operators f_k commutes. For any subset $J = \{i_1, \dots, i_p\}$ of $\{1, \dots, \ell\}$, the weight of $f_{i_1} \dots f_{i_p} \gamma$ is $\mu_{\ell+1} \sigma_{i_p} \dots \sigma_{i_1} - \mu_0$.

Let $x = x_0 p_\mu \in W_e$ where $(x_0, \mu) \in W_0 \times P$ and let $bt_n \dots t_1$ be an expression of x such that $b \in \Pi$ and $t_i \in S$ for all i . Let $(A, \mu_0) \in \text{Alc}_e(\mathcal{F})$. To the reduced expression of x and the alcove (A, μ_0) we associate a gallery $\gamma = (\mu_0, A, F_1, A_1, \dots, F_n, A_n, \mu_{n+1})$ such that $A_i = t_i \dots t_1 A$, F_i the common facet of A_i and A_{i+1} for all $1 \leq i \leq n$ and μ_{n+1} is such that $x(A, \mu_0) = (A_n, \mu_{n+1})$. We denote by $\gamma_{-\tau}$ the gallery associated to our reduced expression of $p_{-\tau}$ and the alcove $(A_0, 0)$. If J is an admissible subset we denote by γ_J the gallery associated to our reduced expression of p_τ and the alcove $(v_J A_0, 0)$. Finally, let \sharp be the involution on the set of galleries defined by reading γ backward and by translating all of its components by $-\mathbf{w}(\gamma)$.

Proposition 6.2. Let $J = \{i_1, \dots, i_p\}$ be a admissible subset and let $J^\dagger = \{j_1, \dots, j_p\}$ be the corresponding LP-admissible subset. We have

$$(f_{j_1} \dots f_{j_p} \gamma_{-\tau})^\sharp = f_{i_1} \dots f_{i_p} \gamma_J.$$

Proof. The gallery $f_{j_1} \dots f_{j_p} \gamma_{-\tau}$ is completely determined by the action of $p_{-\tau}^{J^\dagger}$ on $(A_0, 0)$. If we denote by μ its weight, we have

$$\begin{aligned} p_{-\tau}^{J^\dagger} A_0 &= A_{-\tau} \sigma_{\beta'_{j_p}, N'_{j_p}} \dots \sigma_{\beta'_{j_1}, N'_{j_1}} \\ &= A_\mu \sigma_{\beta'_{j_p}} \dots \sigma_{\beta'_{j_1}} \\ &= A_\mu \sigma_{\beta_{i_1}} \dots \sigma_{\beta_{i_p}} \\ &= v_J A_\mu. \end{aligned}$$

Since $p_\tau^J p_{-\tau}^{J^\dagger} A_0 = A_0 = p_\tau^J v_J A_\mu$, we get the result. \square

We will denote by $\mu(J)$ the weight of the gallery $f_{i_1} \dots f_{i_p} \gamma_J$. According to the proposition above, it is equal to $-\mathbf{w}(f_{j_1} \dots f_{j_p} \gamma_{-\tau})$.

Lenart and Postnikov [9, Corollary 8.3] showed that in a simple Lie algebra with Weyl group W_0 , we have for $\lambda, \tau \in P^+$

$$s_\tau s_\lambda = \sum_J s_{\lambda + \mu(J)}$$

where the sum is over all LP-admissible subsets $J = \{j_1, \dots, j_p\}$ that are λ -dominant and where $-\mu(J)$ is the weight of the gallery $f_{j_1} \dots f_{j_p} \gamma_{-\tau}$. Using the previous proposition together with expression of $P(\tau)C_{p_\lambda w_0}$ obtained at the end of the last section in the case of equal parameters, we get

$$\mathbf{P}(\tau)C_{p_\lambda w_0} = \sum_J C_{p_\tau^J v_J p_\lambda w_0} = \sum_J C_{p_{\lambda+\mu(J)} w_0}.$$

Remark 6.3. The condition at the end of Corollary 8.3 in [9] is not formulate in the same fashion but can be seen to be equivalent.

7 Lakshmibai-Seshadri paths

In this last section, we explain following [9, §9], how to construct an Lakshmibai-Seshadri path (LS paths for short) from an admissible subset J . There is no new result in this section but this construction can be helpful in order to understand the proof of Theorem 5.4.

A rational Ω -path of shape $\mu \in P$ is a pair of sequences $(\underline{\sigma}, \underline{a})$ such that $\underline{\sigma} = (\sigma_0, \sigma_1, \dots, \sigma_r)$ is a strictly decreasing chain (in the Bruhat order) of distinguished left coset representatives of $\text{Stab}_\mu(\Omega_0)$ and $\underline{a} = (a_0, a_1, \dots, a_{r+1})$ is an increasing sequence of rational numbers such that $a_0 = 0$ and $a_{r+1} = 1$.

We identify π with the path $\pi : [0, 1] \longrightarrow V$ defined by

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1}) \mu \sigma_i + (t - a_{j-1}) \mu \sigma_j \text{ for } a_{j-1} \leq t \leq a_j.$$

The weight of π is equal to $\pi(1)$. We set $\pi^* := (\underline{\sigma}^*, \underline{a})$ where $\underline{\sigma}^* := (\sigma_0^*, \sigma_1^*, \dots, \sigma_r^*)$. The path π^* is then of weight $\pi(1)\sigma_{\Omega_0}$.

A rational Ω -path of shape μ is called a Lakshmibai-Seshadri path if there exists a sequence of positive roots $(\delta_1, \dots, \delta_r)$ such that

$$\sigma_0 > \sigma_1 = \sigma_0 \sigma_{\delta_1} > \sigma_2 = \sigma_0 \sigma_{\delta_1} \sigma_{\delta_2} > \dots > \sigma_r = \sigma_0 \sigma_{\delta_1} \dots \sigma_{\delta_r},$$

$\ell(\sigma_i) = \ell(\sigma_{i-1}) - 1$ and $a_i \langle \mu \sigma_i, \delta_i^\vee \rangle \in \mathbf{Z}$.

Remark 7.1. Our definition of LS paths is slightly different from the one of Littelmann in [10] but nearly obviously equivalent. One only needs to notice that if $a = a_q = a_{q+1} = \dots = a_{q+r}$ then the chain $(\sigma_q, \dots, \sigma_{q+r})$ is an a -chain as defined by Littelmann.

For the rest of this section we will need to work with a specific reduced expression of p_τ . The reason for this choice will become clear later. Fix a total order on the set of simple roots $\{\alpha_1, \dots, \alpha_d\}$ and write $\{\omega_1, \dots, \omega_d\}$ for the corresponding fundamental weights. We define the map

$$\begin{aligned} \mathbf{h} : H(A_0, A_\tau) &\longrightarrow \mathbf{R}^{d+1} \\ H_{\delta, k} &\longmapsto \frac{1}{\langle \tau, \delta^\vee \rangle} (k, \langle \omega_1, \delta^\vee \rangle, \dots, \langle \omega_d, \delta^\vee \rangle) \end{aligned}$$

The lexicographic order on \mathbf{R}^{d+1} induces a total order on the set $H(A_0, A_\tau)$. Let

$$H(A_0, A_\tau) = \{H_{\beta_1, N_1}, \dots, H_{\beta_n, N_n}\}$$

be such that $H_{\beta_i, N_i} < H_{\beta_{i+1}, N_{i+1}}$ for all i . Then there exists a reduced expression of p_τ of the form $as_n \dots s_1$ where $a \in \Pi$, $s_i \in S$ and such that H_{β_i, N_i} is the unique hyperplane separating $s_i \dots s_1 A_0$ and $s_{i-1} \dots s_1 A_0$.

Let $J = \{i_1, \dots, i_p\}$ be an admissible subset. For all $k \in \{0, \dots, p\}$, we set (compare to the construction following Proposition 5.5)

- * $J_k = \{i_{k+1}, \dots, i_p\}$ where by convention $J_p = \emptyset$,
- * $\sigma_{J_k} = \sigma_{\beta_{i_{k+1}}} \dots \sigma_{\beta_{i_p}}$ and $v_{J_k} \in W_0$ is such that $v_{J_k} A_0 = A_0 \sigma_{J_k}$,
- * $\gamma_{i_k} = \beta_{i_k} \sigma_{J_k}$ for $k \geq 1$ so that $\sigma_{J_k} = \sigma_{\gamma_{i_p}} \dots \sigma_{\gamma_{i_{k+1}}}$,
- * $\tau_k = \tau \sigma_{J_k}$,
- * $\mu_0 = 0$ and $\mu_k = \mu_{k-1} \sigma_{\gamma_{i_1}} \dots \sigma_{\gamma_{i_k}}$,

* π_k is the straight path defined by

$$\begin{aligned} \pi_k : [0, 1] &\longrightarrow V \\ t &\longmapsto \mu_k + t \cdot \tau_k \end{aligned} ,$$

$$* a_k = \frac{N_{i_k}}{\langle \tau_{k-1}, \gamma_{i_k}^\vee \rangle} = \frac{N_{i_k}}{\langle \tau, \beta_{i_k}^\vee \rangle}.$$

Finally, we set $\underline{\sigma} = (\sigma_{J_0}, \dots, \sigma_{J_p})$ and $\underline{a} = (a_0, \dots, a_{p+1})$ where $a_0 = 0$ and $a_{p+1} = 1$. By construction, the path $\pi = (\underline{\sigma}, \underline{a})$ coincide with the path π_k for all $a_k \leq t \leq a_{k+1}$. Then according to [9, §9] one can show the following result where once again, we assume that we are in the equal parameter case. The choice of our specific reduced expression plays a crucial role in the proof of Statement 3). Recall the definition of $\mu(J)$ in the previous section.

Theorem 7.2. *Let J be an admissible subset. Let $\lambda \in P^+$ and $\pi = (\underline{\sigma}, \underline{a})$ be the Ω -rational path defined above. We have*

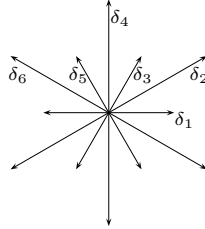
- 1) π is a LS-path of shape τ and weight $\mu(J)$.
- 2) The set of LS paths of shape τ is in bijection with the set of admissible subsets.
- 3) π is λ -dominant (i.e. $\lambda + \pi(t)$ lies in the closure of \mathcal{C}_0 for all $t \in [0, 1]$) if and only if the set of alcoves $p_\tau^J[k, 1]v_J A_\lambda \in \mathcal{C}_0$ for all $k \in \{1, \dots, n\}$.

As a direct consequence of this theorem and of Theorem 5.4, we obtain for $\lambda, \tau \in P^+$

$$\mathbf{P}(\tau)C_{p_\lambda w_0} = \sum_{\pi} C_{p_{\lambda + \pi(1)w_0}}$$

where the sum is over all λ -dominant LS-paths of shape τ .

Example 7.3. Let $\Phi^+ = \{\delta_1, \dots, \delta_6\}$ be a root system of type G_2 :



The associated simple system is $\{\delta_1, \delta_6\}$. The Weyl group of type G_2 is generated by σ_{δ_1} and σ_{δ_6} and the affine Weyl group of type \tilde{G}_2 is generated by $\{\sigma_{\delta_1}, \sigma_{\delta_6}, \sigma_{\delta_3,1}\}$. We'll denote the corresponding element of S by $\{t_1, t_2, t_3\}$. Let τ be the fundamental weight associated to the root δ_6 . In this case, we have $\tau = \delta_4$. It is shown in [8, §18] that

$$t_\tau = \sigma_{\delta_3,1} \sigma_{\delta_5,1} \sigma_{\delta_4,1} \sigma_{\delta_3,1} \sigma_{\delta_5,2} \sigma_{\delta_2,2} \sigma_{\delta_3,2} \sigma_{\delta_4,2} \sigma_{\delta_5,3} \sigma_{\delta_6,1} \quad \text{and} \quad p_\tau = t_2 t_1 t_2 t_1 t_2 t_3 t_1 t_2 t_3.$$

Following [8, Example 10.2], we know that there are 14 admissible subsets. We describe these sets in the table below. In the column saturated chain, we only put the extremal element and one can recover the full chain by adding to the chain all the elements in the columns above: for instance, the saturated chain associated to the admissible subset $\{3, 9, 10\}$ is $\sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6} > \sigma_{\delta_5} \sigma_{\delta_6} > \sigma_{\delta_6} > e$.

Saturated chains	reduced expression	admissible subset
1	1	\emptyset
σ_{δ_6}	t_2	$\{10\}$
$\sigma_{\delta_5} \sigma_{\delta_6}$	$t_2 t_1$	$\{9, 10\}, \{5, 10\}, \{2, 10\}$
$\sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6}$	$t_2 t_1 t_2$	$\{8, 9, 10\}, \{3, 9, 10\}, \{3, 5, 10\}$
$\sigma_{\delta_3} \sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6}$	$t_2 t_1 t_2 t_1$	$\{7, 8, 9, 10\}, \{4, 8, 9, 10\}, \{1, 8, 9, 10\}, \{1, 3, 9, 10\}, \{1, 3, 5, 10\}$
$\sigma_{\delta_2} \sigma_{\delta_3} \sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6}$	$t_2 t_1 t_2 t_1 t_2$	$\{6, 7, 8, 9, 10\}$

We now compute the different elements needed in the construction of the path π_J where $J = \{3, 5, 10\}$.

k	J_k	$\beta_{i_k}^*$	γ_{i_k}	σ_{J_k}	τ_k	N_{i_k}	a_k
0	$\{3, 5, 10\}$			$\sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6}$	$-\delta_6$		0
1	$\{5, 10\}$	δ_4	δ_6	$\sigma_{\delta_5} \sigma_{\delta_6}$	δ_6	1	1/2
2	$\{10\}$	δ_5	δ_1	σ_{δ_6}	δ_2	2	2/3
3	\emptyset	δ_6	δ_6	1	δ_4	1	1

The LS path associated to J is represented in red in Figure 1: we see that

- * it follows the direction $\delta_4 \sigma_{\delta_4} \sigma_{\delta_5} \sigma_{\delta_6} = -\delta_6$ for a time 1/2;
- * it follows the direction $\delta_4 \sigma_{\delta_5} \sigma_{\delta_6} = \delta_6$ for a time 1/6;
- * it follows the direction $\delta_4 \sigma_{\delta_6} = \delta_2$ for a time 1/3.

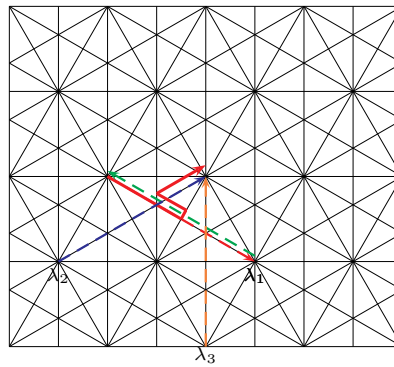


Fig. 1: LS path associated to the set $J = \{3, 5, 10\}$.

When doing it for all J we obtain 14 paths corresponding to the 14 admissible subsets as shown in Figure 2.

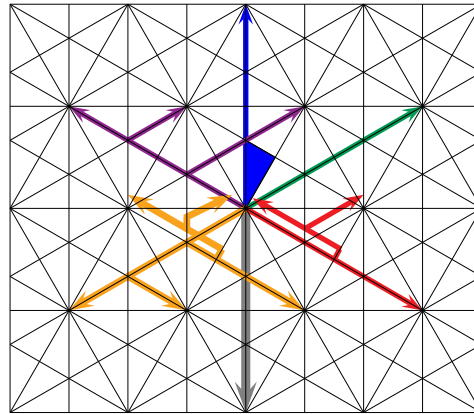


Fig. 2: LS paths in type G_2 .

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